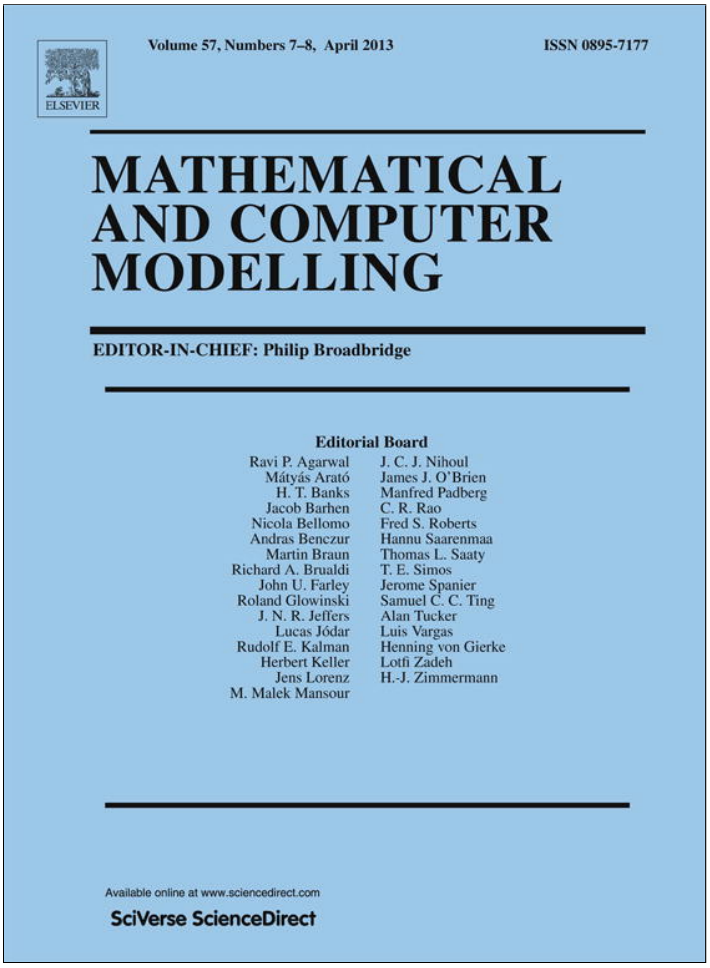

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Sentimental equilibria with optimal control

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ABSTRACT

The ubiquitous phenomenon of marital dissolution in the West poses a major theoretical problem that is not well understood. The fact that most couples declare that they are committed to an enduring and happy relationship allows us to model their sentimental relationship as a simple optimal control problem. A state variable monitors the wellness of the relationship whose natural decay in time must be counteracted with effort – the control variable – according to a widely accepted principle in marital psychology. Equilibria are basic desirable solutions of this control problem. In this note key properties of equilibria – in particular, existence, dynamical instability and sensitivity analysis – are obtained for a class of decaying sentimental dynamics. An underlying mechanism combining the existence of an effort stress with the instability of the model may be identified as a main cause of distress for the couple relationship.

1. Introduction

There is an epidemic of failure among couple relationships in the West. In 1989 demographers Martin and Bumpass [1] predicted that two thirds of all US marriages would end up in separation or divorce within a 40-year span. At present the divorce rate in the US could be at about 50%. The same phenomenon occurs in western countries. In the European Union, there is a marital breakdown every 30 s. The last available data shows that the divorce rate in EU27 is about 0.43. In Spain there are 2 divorces for every 3 marriages [2]. In general cohabitation is less stable than marriage [3].

A second piece of evidence concerns the attitudes of each partner, rather than demographic data. First, people typically consider that a stable relationship is a main ingredient of happiness. This has been evidenced in different surveys. In Spain, for instance, most college students in a recent survey declared that they absolutely want to have a stable relationship [4]. Second, most people consider their own relationship stable. Also in Spain, the majority of respondents in another survey qualified as definite their project of life in common with their partners [5].

The two facts above pose together what is called in [6] *the failure paradox*: how is it that couples that are ready to commit to a lasting happy relationship will probably fail? The failure paradox is related to an old problem in marital psychology: why are some marriages happy while others are miserable? Gottman et al. [7] claim that mathematical theory is desperately needed in the field of marital research, in particular to answer the previous question. In their pioneer work [7], they develop a model to analyze marital instability, by calibrating a system of difference equations for the evolution of partners' emotions during a conversation. The interaction dynamics they prescribe is descriptive, based on experience and their intuitive understanding of the influence among partners during conflict.

In [6] a different mathematical modeling approach is proposed to analyze marital dynamics. It is based on a behavioral assumption: couples design their relationships to last forever with the objective to be as happy as possible. Notice that this seems consistent with the sociological evidence quoted above. According to a widely accepted principle in the psychology

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of marital relationships, marriages starts off happy but over time a reinforcement erosion occurs that is the source of marital dysfunction [8]. Also, the erosion can be counterbalanced with effort, in the form of conscious practices to keep the relationship alive and well (see e.g. [7]). Thus the couple's problem can be naturally seen as an optimal control problem where marital status – measured in terms of attachment or feeling – is the state variable and effort making is the control variable.

Sentimental equilibria – stationary feeling levels sustained by a routine effort – are considered key solutions for the control problem. Assuming that the feeling variable decays at a constant rate, it is shown in [6] that a unique sentimental equilibrium exists. It is also proved that the equilibrium is of saddle type – in turn unstable – and such that there exists an effort gap, that is, the effort level at equilibrium is always higher than the favorite effort level defined by the couple a priori. These two facts were identified as a basic underlying mechanism that may lie behind the distress and eventual disruption of many sentimental relationships that were initially planned to last forever. In turn, the model findings may account for the failure paradox.

In this note, the simple model in [6] is further explored. Assuming a more general decaying law for the feeling, the key issues of existence and instability of equilibria – as well as the presence of the effort gap – are considered. The sensitivity of equilibria with respect to changes in the significant parameters of the model is also analyzed.

2. The model: optimal control of sentimental dynamics

The basics of the theory in [6] are summarized below. The model builds on three assumptions.

(A1) (*Weak homogamy*) The couple itself is the planning unit.

Therefore, no interaction dynamics within the couple is considered. This strong assumption can be seen equivalent to considering the partners as very similar. In psychology this assortative mating is called *homogamy* – marriage between individuals who are, in some culturally important way, similar to each other. In Western societies the most frequent form of assortative mating is a strong form of homogamy [9]. In the model much less similarity is needed: partners just share the parameters and functional structure defined below.

(A2) (*The "second law"*) There is a natural fading inertia for the sentimental dynamics that can be counteracted with effort.

Conventional wisdom says that "love is not enough"; therapists give a long list of things that can be done in order to maintain a strong relationship. This fact was formulated in [7] as the *second law of thermodynamics for marital relationships*.

Assume that the state of the relationship at time $t \geq 0$ is monitored by $x(t) \geq 0$ – the feeling variable – which is a measure of marital satisfaction or the couple's common feeling about the relationship. The level of effort – injecting "energy" into the relationship – is given by the variable $c(t) \geq 0$ – the effort variable. If the feeling decay is steered by a non-negative smooth (C^2) function $h(x)$, the second law can be written as

$$\frac{dx}{dt} = -h(x) + ac, \tag{1}$$

where $a > 0$ is a parameter associated with effort efficiency. It is assumed that, for $x \geq 0$,

$$h(x) > 0, \qquad h(0) = 0, \qquad h'(x) \geq 0, \qquad h'(0) < +\infty, \quad \text{and} \quad h(x) \to +\infty \quad \text{as } x \to +\infty. \tag{2}$$

The natural choice $h(x) = rx$, with $r > 0$, corresponds to the linear case considered in [6]. In this case, without effort, the feeling decays at a constant rate given by the parameter $r > 0$. Since setting $c = 0$ in (1) implies that

$$\frac{d^2x}{dt^2} = h(x)h'(x),$$

assuming $h'(x) \geq 0$ implicitly entails that, with no effort forcing, $x(t)$ decays in time in a convex fashion. Thus the assumptions in (2) form a natural extension of the linear case. Notice that (2) includes the case that $h'(x) = 0$ for x in some interval, where $x(t)$ in turn would vanish linearly in time. The assumption about the asymptotic behavior of h can be relaxed; it is enough that $h(x)$ grows beyond a certain level (see below). It is assumed that there exists a threshold feeling level $x_{\min} > 0$ below which the relationship is not considered satisfactory or viable.

While $x(t)$ cannot be altered consciously, the effort $c(t)$ is a rational variable; its intensity can be regulated so that $x(t)$ does not collapse and, furthermore, some goal is achieved. This is the setting of an optimal control problem, with $x(t)$ the state variable and $c(t)$ the control variable of the problem. Control theory requires some regularity on these variables, namely $c(t)$ is assumed piecewise continuous, so that in turn $x(t)$ is piecewise C^1. These assumptions fit well within the framework of the sentimental model for, while gaps in the effort may be reasonably expected to occur at some points – so that $c(t)$ will have a jump discontinuity at such points – feeling is plausibly expected to vary smoothly, except at a point of effort discontinuity.

(A3) (*Valuation of well-being*) Couples asses their well-being in a cost-benefit fashion.

Feeling is something good that produces satisfaction or utility but in a decreasing way, which is a standard idea in psychology. So feeling satisfaction is described by some increasing and concave smooth function $U(x)$, defined for $x \geq 0$, that gets flat when feeling becomes large:

$$U'(x) > 0, \qquad U''(x) < 0, \qquad U'(x) \to 0 \quad \text{as } x \to +\infty. \tag{3}$$

High levels of effort are considered a bad thing. Since bad things escalate, according to a well known principle in psychology [10], discomfort is increasing and convex as a function of effort. In the model, however, effort making is not assumed to produce unwellness from the very first unit. Rather there is a key effort level $c^* \geq 0$ which is preferred a priori. If $D(c)$ gives discomfort or disutility as a smooth function of effort $c \geq 0$, it is assumed that

$$D''(c) > 0, \qquad D'(c^*) = 0, \qquad D'(c) \to +\infty \quad \text{as } x \to +\infty. \tag{4}$$

Therefore, effort levels below c^* have in fact a positive effect on well-being; partners enjoy making the effort provided it is not too much. This positive effect reaches its maximum at c^*, and then beyond c^* effort making actually produces disutility. Because of convexity, from level c^* on, each extra unit of effort produces more and more discomfort.

Assuming (A1)–(A3), the couple's control problem consists of determining the required effort path so that total happiness is maximal and the feeling-effort dynamics is sustainable in the long-term. Total well-being is expressed as

$$\int_0^\infty \exp(-\rho t)\,(U(x) - D(c))\,dt,$$

which gives the total discounted value of net well-being, measured as the difference between feeling utility and effort disutility. The upper limit is infinite because the relationship is designed to last forever. The exponential weight – the standard discount term used in economics – represents temporal preference: future utility is less valuable than current utility. The parameter $\rho > 0$ is the discount factor.

3. Results and discussion

The current-value Hamiltonian of the optimal control problem defined above is given by $H(x, c, m) = U(x) - D(c) + m(-h(x) + ac)$, where $m(t)$ is the current-value co-state variable. See e.g. [11] for the basic results of optimal control theory. For $c(t) > 0$ optimal, Pontryagin's maximum principle implies that $D'(c(t)) = am(t)$. It follows that $\frac{dc}{dt}D''(c) = a\frac{dm}{dt}$ wherever $c(t)$ is differentiable. The co-state equation is $\frac{dm}{dt} = -U'(x) + m(\rho + h'(x))$. As a consequence, the optimal effort must satisfy

$$\frac{dc}{dt} = \frac{1}{D''(c)}\left(D'(c)(\rho + h'(x)) - aU'(x)\right). \tag{5}$$

This key equation provides an optimal behavioral rule in terms of the effort variable.

This note is concerned about optimal sentimental equilibria, that is, stationary solutions $(\bar{x}, \bar{c})$ of the feeling-effort dynamical system (1)–(5). Equilibria must solve

$$\begin{cases} 0 = -h(x) + ac, \\ 0 = D'(c)(\rho + h'(x)) - aU'(x). \end{cases} \tag{6}$$

Existence of an effort gap. It follows from (2), (3) and (6) that any equilibrium $(\bar{x}, \bar{c})$ satisfies

$$D'(c) = \frac{aU'(x)}{\rho + h'(x)} > 0. \tag{7}$$

Then, $D'(\bar{c}) > 0$ and (4) imply that $\bar{c} > c^*$. As a consequence, there is an effort gap in equilibrium: the required effort level is higher than the favorite level. This can be seen as a primary source of instability: the discomfort entailed by the effort gap may be felt unbearable by the partners. Even if the equilibrium feeling is rewarding (i.e. $\bar{x} > x_{\min}$), the relationship may not be viable due to a large effort gap.

Existence of equilibria. Equilibria are located at the intersection of the vertical and horizontal nullclines, defined by the equations in (6). From (2) and (4), it holds that $(\rho + h'(x))D''(c) \neq 0$ for all $x \geq 0$, and then the horizontal nullcline $0 = D'(c)(\rho + h'(x)) - aU'(x)$ can be described by a smooth function $c = c_H(x)$ in the (x, c)-plane. Since (7) holds for $c = c_H(x)$, it follows that $c_H(x) > c^*$ for all $x \geq 0$. Also, using (2) and (3), (7) also implies that $D'(c_H(x)) \to 0$ as $x \to +\infty$ and in turn that $c_H(x) \to c^*$ as $x \to +\infty$. From (2), a continuity argument ensures that nullclines intersect at least once in the positive quadrant. Therefore, an equilibrium exists for the feeling-effort dynamics. Notice that the same conclusion holds if $h(x)$ is not unbounded as $x \to +\infty$ but it grows beyond level c^*.

Under the extra assumption that h is convex, the equilibrium is furthermore unique. In that case, the nullclines cross just once, because $c_H(x)$ is decreasing. This can be checked by differentiating the equation of the horizontal nullcline to obtain

$$c_H'(x) = \frac{aU''(x) - D'(c_H(x))h''(x)}{(\rho + h'(x))D''(c_H(x))}. \tag{8}$$

From (2)–(4), this expression is negative if $h'' \geq 0$. It also remains negative in the case that h changes its curvature provided that it does not bend too much in the concave region.

Instability and local dynamics. Let $(\bar{x}, \bar{c})$ be an equilibrium of the feeling-effort dynamics. Using (6) and (8), the Jacobian of the system (1)–(5) at $(\bar{x}, \bar{c})$ can be written as

$$J(\bar{x}, \bar{c}) = \begin{pmatrix} -h'(\bar{x}) & a \\ -(\rho + h'(\bar{x}))c_H'(\bar{x}) & \rho + h'(\bar{x}) \end{pmatrix}. \tag{9}$$

Since trace$J(\bar{x}, \bar{c}) = \rho > 0$, the equilibrium is unstable. Notice that this is independent of the uninfluenced dynamics h of the feeling or the existence of the effort gap: whenever there is a sentimental equilibrium, it is unstable.

Since $\text{sign}(\det J(\bar{x}, \bar{c})) = \text{sign}(ac_H'(\bar{x}) - h'(\bar{x}))$, the local dynamics near an equilibrium is determined by the way nullclines intersect. A saddle-type dynamics occurs if the slopes of the nullclines satisfy $c_H'(\bar{x}) < \frac{1}{a}h'(\bar{x})$. In particular, if $c_H(x)$ is non-increasing – in turn if h is convex – the unique equilibrium is a saddle.

Myopic equilibria. Setting $c(t) = c^*$ in the state equation (1) could be considered a natural – convenient – way to sustain the sentimental relationship. Since the corresponding state dynamics is stable, in the long-term the feeling is driven to the feeling level x^* defined by the equation $c^* = \frac{1}{a}h(x^*)$. That is, however, a poor solution based on a short-term (myopic) approach of minimizing the cost of effort. To see why, first notice that the solution (x^*, c^*) is sub-optimal: it is apparent from the effort Eq. (5) and properties (3)–(4) that $c = c^*$ is never a solution of the optimal control problem. Also, from (2), the fact that $c^* < c_H(x)$ for all $x \geq 0$ implies that the vertical nullcline intercepts first the line $c = c^*$ and then the horizontal nullcline. This in turn means that $x^* < \bar{x}$. Then, if $c = c^*$, the required condition $x - x_{\min} > 0$ for an admissible relationship may no longer hold – the gap $x - x_{\min}$ anyway worsens, what may endanger the continuation of the relationship.

Sensitivity analysis. To understand how equilibrium responds to small changes in the parameters of the model, consider the linear case $h(x) = rx$, $r > 0$ – it was not analyzed in [6]. Since $h'' = 0$, the unique equilibrium is a nonlinear saddle in this case. Since

$$\det \begin{pmatrix} -r & a \\ -aU''(\bar{x}) & (\rho + r)D''(\bar{c}) \end{pmatrix} < 0, \tag{10}$$

the equilibrium $(\bar{x}, \bar{c})$ is a smooth function of the parameters (r, a, ρ) defined through the system (6). Differentiating (6), the sign of the partial derivatives of $\bar{x}$ can be easily determined from (3), (4), (7) and (10). Since $\bar{c} > c^*$, it follows that

$$\frac{\partial \bar{x}}{\partial r} < 0, \qquad \frac{\partial \bar{x}}{\partial a} > 0, \qquad \frac{\partial \bar{x}}{\partial \rho} < 0.$$

It can also be determined unambiguously that $\frac{\partial \bar{c}}{\partial \rho} < 0$. The responses of $\bar{c}$ to a small variation in r or in a are more subtle. Using (7), differentiating (6) gives

$$\text{sign}\, \frac{\partial \bar{c}}{\partial r} = \text{sign}\left(-r\frac{U'(\bar{x})}{\rho + r} - \bar{x}U''(\bar{x})\right).$$

Thus $\frac{\partial \bar{c}}{\partial r} > 0$ if and only if $\frac{-\bar{x}U''(\bar{x})}{U'(\bar{x})} > \frac{r}{r+\rho}$. The first term in this expression is the elasticity of the marginal utility at equilibrium $\text{El}_{\bar{x}}(U')$ (see e.g. [12]). It follows that

$$\text{sign}\, \frac{\partial \bar{c}}{\partial r} = \text{sign}\left(\text{El}_{\bar{x}}(U') - \frac{r}{r + \rho}\right).$$

Similarly, using (6), it can be obtained that

$$\text{sign}\, \frac{\partial \bar{c}}{\partial a} = \text{sign}\,(1 - \text{El}_{\bar{x}}(U')).$$

The analysis implies that, in equilibrium, a small increase in the discount factor may endanger the viability of the relationship, since the equilibrium feeling decreases. On the other hand, effort at equilibrium gets closer to the favorite effort c^*, that eases the effort to sustain equilibrium. As a result of both effects, a small decrease in the valuation of future happiness – other things being equal – makes the relationship less rewarding but more comfortable. The effects on the quality of the relationship caused by a small variation in either the rate r of decay of feeling or the efficiency a of effort can be obtained in a similar way. In those cases, the discussion about the effect caused on the effort depends on the elasticity of the marginal utility at equilibrium – that is, the response of marginal utility to a one percent increase in feeling.

4. Concluding remarks

Human relationships usually pose difficult problems that may appear very complex when sentiments are a driving component of the interaction. This seems particularly true in the case of romantic relationships. Given the high failure rates among couple relationships in the West, understanding what makes an enduring and happy couple is a substantial issue. The fact that couple break-up is such a pervasive phenomenon suggests that a simple deterministic mechanism may be acting behind many sentimental failures. Long-term relationships, like marriages, are fueled by a complex mix of emotions – notably romance, lust or attachment, among others [13]. These are amalgamated in a unique overall feeling, which is represented by the (time dependent) feeling variable $x(t)$ in this paper. According to psychological evidence, the level of the fuel variable $x(t)$ naturally decays in time and must be recovered through effort. This fact is combined with a behavioral rule – couples intend to stay together to be as happy as possible – to modeling a couple's relationship as a simple optimal control problem. Solving the control problem determines the required effort levels to sustain the relationship in the long term. The model simplifies by considering homogamous couples, composed by partners that share feeling and effort levels and have the same valuation scheme about their relationship.

In terms of individual behavior, the symmetry due to the homogamy assumption implies that each partner solves the same problem. As a consequence, when solving his problem a partner can think of $x(t)$ as the feeling of the other rather than his own. His mate's feeling is thus a main source of his satisfaction. This interpretation matches with a more interactive – disinterested – form of love. Disinterested love, according to Leibniz, consists of "not thinking about or asking for any pleasure of one's own except what one can get from the happiness or pleasure of the loved one" [14]. Of course, disinterested love truly produces an interesting case study when both partners differ in the variables or in the utility structure of the model. This is the case of heterogamous couples, which requires an extension of the optimal control model in this paper. In Western society human mate choice is based on a strong form of homogamy.

Within the simplified setting described above, optimal control theory provides a natural framework to model long-term sentimental relationships. In particular, this paper is concerned with sentimental equilibria – stationary solutions for the couple's control problem. These are particularly appealing, either as states of relationships already settled or as sustainable destinies in the long-term for newlywed couples. For a class of diminishing laws for the feeling, the mathematical analysis reveals a basic obstacle for the sustainability of a relationship in equilibrium, namely the existence of an effort gap along with dynamical instability. The mechanism goes as follows. First, to sustain equilibrium, a couple must fine-tune the correct effort only to find that is bigger than the effort level that they prefer. This may conceivably lead to relaxing the required effort level. Then, local instability operates against restoring things back to equilibrium.

A natural way out of the pessimistic scenario above consists of assuming some resilience in the feeling to degrade. This can be introduced in the model by considering a different class of dynamics for the feeling, e.g. consistent with decaying in a concave fashion for large levels and in a convex way at lower levels. The analysis in this note shows that this kind of resilience might avoid the effort gap at equilibrium but not the instability. A sentimental equilibrium with optimal control is unstable regardless of the decaying dynamics of the model.

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